DESCRIPTIVE SOLUTION OF THE INVERSE HEAT-CONDUCTION PROBLEM IN THE B-SPLINE BASIS

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The article presents a stable algorithm of solving an integral equation of the first kind taking into account the a priori information on the sought solution.

Statement of the Problem. Some statements of the inverse heat-conduction problem lead to the solution of an integral equation of the first kind

$$\int_{c_x}^{d_x} K(y,\ x)\,\varphi(x)\,dx = f(y),\quad y\in[c_y,\ d_y], \tag{1}$$

which is an incorrectly stated problem [1]. It is characteristic of many regular (stable) methods of solving this class of problems that we can use a priori information on the smoothness of the sought solution $\varphi(x)$ specified in different ways. Moreover, in many cases researchers have additional qualitative information in regard to $\varphi(x)$ that characterizes different properties of the sought solution (monotonic increase, decrease, concaveness, convexity, positiveness, boundary conditions, etc.).

In the present work we introduce such a priori information on $\varphi(x)$ by a system of inequalities

$$\varphi^{(l_i)}(x_i^*)\left\{\begin{matrix}\leqslant\\ =\\ \geqslant\end{matrix}\right\} d_i,\quad i=1,\ \ldots,\ N_r, \tag{2}$$

where $\varphi^{(l_i)}(x_i^*)$ is the derivative of l_i-th order of the function $\varphi(x)$ at the point x_i^*, and as the relation we may take any of the three relations in the braces. According to the registered values of the right-hand side $\tilde{f}_i = f(y_i)+\xi_i,\ i=1,\ \ldots,\ n,$ we have to construct a solution satisfying (2). Since the constraints of (2) are often of a qualitative descriptive nature (i.e., larger or smaller than the specified value), we call in conformity with [2] such a solution a descriptive solution of Eq. (1).

Determination of the Domain of Descriptive Solutions. Let us examine the problem of selecting the domain whose elements will approximate the solution of the integral equation (1). It is expedient to take the domain $P_{m,z,\nu}$ of the piecewise polynomials of m-th degree [3]. For its determination we introduce two sequences: the strictly increasing sequence of real numbers $z=\{z_1,\ z_2,\ \ldots,\ z_{N_z}\}$, where $z_1\leqslant c_x,\ z_{N_z}\geqslant d_x$, and the sequence of positive integers $\nu=\{\nu_1,\ \nu_2,\ \ldots,\ \nu_{N_z}\}$. Then, if the function $\psi(x)\in P_{m,z,\nu}$, then

$$\psi(x)=P_i(x),\quad \text{if}\quad z_i\leqslant x\leqslant z_{i+1};$$
$$\text{jump}_{z_i}\left(\frac{d^k\psi(x)}{dx^k}\right)=0,\quad \begin{matrix}k=0,\ \ldots,\ \nu_i-1;\\ i=2,\ \ldots,\ N_z-1,\end{matrix}$$

where $P_i(x)$ is a polynomial of m-th degree with real coefficients; $\text{jump}_{z_i}\psi(x)=\psi(z_i^+)-\psi(z_i^-)$ is the "jump" of the function $\psi(x)$ at the point $x = z_i$. Approximation of the solution by

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elements of the domain $P_{m,z,\nu}$ yields a good approximation for $\varphi(x)$ (by the corresponding selection of the degree m and of the sequence z) with the required properties of continuity. Thus, if we take $\nu_i = m$, $i = 2, \ldots, N_z - 1$, we obtain a solution with derivatives that are continuous in the interval (z_1, z_{N_z}) up to and including the (m − 1) interval. If necessary, we may take $\nu_i = 0$, and may then obtain a solution with a discontinuity of the first kind at the point $x = z_i$.

An expedient base of the domain $P_{m,z,\nu}$ may be the sequence of normalized B-slines (base splines) of m-th degree [3, 4]. If the nodes $\{\tilde{x}_i\}$, $i = 1, \ldots, N+m+1$ of the B-splines are specified via the elements z_i by the relations

$$\begin{gathered}\tilde{x}_1 \leqslant \tilde{x}_2 \leqslant \ldots \leqslant \tilde{x}_{m+1} = z_1; \quad \tilde{x}_{l(j)} = \tilde{x}_{l(j)+1} = \ldots = \tilde{x}_{l(j+1)-1} = z_j, \\ j = 2, \ldots, N_z - 1; \quad z_{N_z} = \tilde{x}_{N+1} \leqslant \tilde{x}_{N+2} \leqslant \ldots \leqslant \tilde{x}_{N+m+1},\end{gathered} \tag{3}$$

where $l(j) = m + 1 + \sum_{i=2}^{j} (m + 1 - \nu_i)$; $N = m + 1 + \sum_{i=2}^{N_z-1} (m+1-\nu_i)$, then the following statement is correct.

Statement 1 [3]. The sequence of normalized B-splines $B_{1,m}, B_{2,m}, \ldots, B_{N,m}$ of the m-th degree with nodes (3) is the base of the domain $P_{m,z,\nu}$ in the interval (z_1, z_{N_z}).

Consequently, we represent the sought approximation of the solution of $\varphi(x)$ by an element of the domain $P_{m,z,\nu}$ by the combination

$$\varphi_N(x, a) = \sum_{j=1}^{N} a_j B_{j,m}(x). \tag{4}$$

This approximation is fully determined by the coefficients a_j, $j = 1, \ldots, N$, which are coordinates of the element of the domain $P_{m,z,\nu}$ in the base of the B-splines.

Algorithm for Calculating the Coefficients of the Descriptive Solution. The right-hand side of expression (4) may be represented in the form

$$f_N(y) = \sum_{j=1}^{N} a_j R_j(y),$$

where the function $R_j(y) = \int K(y, x) B_{j,m}(x)dx$ is interpreted as the expansion of the kernel of Eq. (1) in the base of the B-splines. With specified parameters m, z, ν of the domain $P_{m,z,\nu}$ the vector $a = \{a_1, \ldots, a_N\}$ of the coefficients of the solution $\varphi_N(x, a)$ can be determined by the least squares method, i.e., from the minimum of the functional

$$\Phi(a) = \sum_{i=1}^{n} p_i (\tilde{f}_i - f_N(y_i))^2$$

with the constraints (2). The weighting factors p_i are strictly positive, and they characterize the significance of the i-th measurement. If we introduce into the examination the symmetric matrix Q with the elements $Q_{kj} = \sum_{i=1}^{n} p_i R_k(y_i) R_j(y_i)$ and the vector q with the projections $q_j = \Sigma p_i \tilde{f}_i R_j(y_i)$, we can represent $\Phi(a)$ in the form of the quadratic functional

$$\Phi(a) = a^{\mathrm{T}} Q a - 2a^{\mathrm{T}} q + \sum_{i=1}^{n} p_i \tilde{f}_i^2. \tag{5}$$

If we denote $c_{ij} = B^{(l_i)}_{j,m}(x^*_i)$, we can write the constraints (2) in the form of a system of equalities and inequalities

$$g_i(a) = \sum_{j=1}^{N} c_{ij}a_j - d_i = 0, \quad i = 1, \ldots, n_r;$$
$$g_i(a) = \delta_i\Big(\sum_{j=1}^{N} c_{ij}a_j - d_i\Big) \leqslant 0, \; i = n_r + 1, \ldots, N_r, \tag{6}$$

where n_r is the number of constraints in the form of equalities; the factor δ_i assumes the value 1 if the i-th constraint has $\leqslant$, and $\delta_i = -1$ if it has $\geqslant$. This system determines in the N-dimensional domain Ω_N of coefficients a_j some permissible domain Ω^d_N whose points satisfy the inequalities (6).

Thus the problem of determining the coefficients of the descriptive solution reduced to the problem of quadratic programming: to find the vector $a^{**} \in \Omega^d_N$ furnishing to the functional (5) the minimum, i.e.,

$$\Phi(a^{**}) = \inf_{a \in \Omega^d_N} \Phi(a). \tag{7}$$

It is natural to assume that the domain Ω^d_N contains at least one point. Otherwise there is no sense in constructing the descriptive solution.

The problem of quadratic programming was solved by the method of penalty functions. Without explaining the method itself (see [5]), we want to point out that the basic idea consists in the approximate reduction of problem (7) to a sequence of minimization problems without constraints on some auxiliary functional. As the latter we took $F(a, \gamma) = \Phi(a) + \gamma G(a)$, where γ is a penalty factor, $G(a)$ is a penalty functional of the form

$$G(a) = \sum_{i=1}^{n_r} \omega_i |g_i(a)|^2 + \sum_{i=n_r+1}^{N_r} \omega_i [\max(0, g_i(a))]^2,$$

which is strictly larger than zero if the vector a does not belong to the permissible domain Ω^d_N, and equal to zero (and consequently, $F(a, \gamma) = \Phi(a)$) if $a \in \Omega^d_N$.

Statement 2 [6]. The minimum of the functional $F(a, \gamma)$ with any fixed magnitude $\gamma \in (0, \infty)$ is attained at the unique point a^*_γ. The limit point a^*_∞ of the sequence a^*_γ with $\gamma \to \infty$ is the solution of the problem (7), and the values of $\Phi(a^*_\gamma)$ increase monotonically, and $\lim \Phi(a^*_\gamma) = \Phi(a^{**})$ when $\gamma \to \infty$.

Statement 3. With fixed parameters m, z, ν and simultaneous constraints (6) there exists a unique descriptive solution $\varphi_N(x, a^{**})$ of Eq. (1) in the domain $P_{m,z,\nu}$.

Selection of the Dimensionality of the Domain of Descriptive Solutions. The dimensionality N (number of base splines) in the given method is the singular parameter of regularization of the obtained solution. It has to be matched with the noise level of the measurement. The dimensionality N_{opt}, which minimizes the rms error of approximating the exact solution $\varphi(x)$ by the element $\varphi_N(x, a)$ among all other values of N with the given noise level, can be determined as the optimum dimensionality of the domain $P_{m,z,\nu}$. Below we present two methods of evaluating N_{opt} with different a priori information on the statistics of the measurement noise.

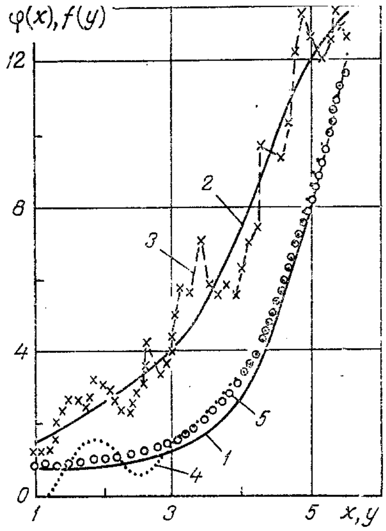

Fig. 1. Results of the computing experiment: 1) exact solution $\varphi(x)$; 2) exact right-hand side f(y) of the integral equation; 3) measured values $\tilde{f}_i$; 4) rms solution $\varphi_N(x, a^*)$; 5) descriptive solution $\varphi_N(x, a^{**})$.

1. It is assumed that the measurement noise ξ_i has a zero mean and the known scatter σ_i^2. We introduce into the examination the bilinear form $R(N)=\sum_{i=1}^{n}\tilde{f}_i e_i(N)/\sigma_i^2$, where $e_i(N)=\tilde{f}_i-f_N(y_i)$ is the discrepancy of the i-th measurement. The value N_R for which

$$R(N_R)\in[\theta_M(\beta/2),\quad \theta_M(1-\beta/2)], \tag{8}$$

is the evaluation of the optimum dimensionality N_{opt}. The boundary points $\theta_M(\beta/2)$, $\theta_M(1-\beta/2)$ of the interval are the quantiles of the χ^2-distribution with $M = n - N + N_A$ degrees of freedom of the levels $\beta/2$, $1 - \beta/2$, respectively; N_A is the number of constraints of system (6) converted by vector a^{**} into identities with specified accuracy. Condition (8) follows from the criterion of optimality of the approximation by experimental information [7].

2. It is assumed that the measurement noise has zero mean and unknown scatter. In this case the evaluation of N_{opt} is the magnitude N_V which furnishes the minimum to the functional

$$V(N)=\frac{1}{n}\sum_{i=1}^{n}e_i^2(N)\Big/\left[1-\frac{N-N_A}{n}\right]^2.$$

This method is a generalization of the method of cross-validation [8] to the case when the sought solution has the form (4).

Program Realization and Results of the Computing Experiment. The explained algorithm for constructing the descriptive solution was realized in the form of a complex of subprograms written in FORTRAN-IV. The complex envisages the detection and diagnosing of errors in the initial data and also of errors arising in the course of the computing process of constructing the descriptive solution. We will dwell on some results of the computing experiment with this complex of subprograms.

The solution of the integral equation with the kernel $K(y, x) = 1/(1 + (y-x)^2)$ was specified by the downward convex function $\varphi(x)=\exp(-(x-10)^2/1000)+20\exp(-(x-7)^2/4)$, $x\in[1; 5.5]$. The values of the right-hand side $f(y_i)$, i = 1, ..., 60, were distorted by the numbers with normal distribution and with scatter $\sigma_i^2=(\varepsilon|f(y_i)|/2.5)^2$, where $\varepsilon = 0.25$ is the relative noise.

From the initial data $\{y_i, \tilde{f}_i, p_i = 1/\sigma_i^2\}$ we constructed two solutions: the descriptive solution $\varphi_N(x, a^{**})$ satisfying the constraints

$$\varphi(1) = 0.92, \quad \varphi''(z_j) \geqslant 0, \quad j = 1, \ldots, N_z; \tag{9}$$

the rms solution $\varphi_N(x, a^*)$ whose coefficients were determined from the minimum of (6) without taking (9) into account. For both solutions the elements of the sequence z were specified by the values {0.5; 1.5; 2.5; 3.5; 4.5; 5.0; 5.5; 6.1}, $N_z = 8$, m = 3. It can be demonstrated that for cubic B-splines the downward complexity of the solution $\varphi_N(x, a^{**})$ in the interval [1; 5.5] follows from the fulfillment of (9). Figure 1 shows the functions $\varphi(x)$, f(y) and the solutions $\varphi_N(x, a^*)$, $\varphi_N(x, a^{**})$. An analysis of this and of other computing experiments shows that if reliable a priori information on the sought solution is taken into account, it greatly improves the accuracy of the solution of an integral equation of the first kind.

NOTATION

$\varphi(x)$, sought solution of the integral equation; K(y, x), kernel of the integral equation; f(y), exact right-hand side of the integral equation; $\varphi^{(l_i)}(x_i^*)$ ($i = 1, \ldots, N_r$), derivative of l_i-th order of the function $\varphi(x)$ at the point x_i^*; d_i ($i = 1, \ldots, N_r$), constraints imposed on the values $\varphi^{(l_i)}(x)$; $\tilde{f}_i$ ($i = 1, \ldots, n$), measured values of the right-hand side of the integral equation; ξ_i, measurement noise; $P_{m,z,\nu}$, domain of the descriptive solutions; $\tilde{x}_i$ ($i = 1, \ldots, N + m + 1$), nodes of the B-spline $B_{j,m}$ of the m-th degree; N, dimensionality of the domain of the solutions; a_j, coefficients of the solution in the base of the B-splines; $\Phi(a)$, quadratic functional; p_i, weighting factors contained in $\Phi(a)$; $g_i(a)$ ($i = 1, \ldots, N_r$), linear form determining the permissible domain Ω_N^d for the coefficients a_j; N_{opt}, optimum dimensionality of the domain of the solutions; $e_i(N)$ ($i = 1, \ldots, n$), discrepancy of the i-th measurement; $\varphi_N(x, a^*)$, rms solution; $\varphi_N(x, a^{**})$, descriptive solution of the integral equation.

LITERATURE CITED

1. A. N. Tikhonov and V. Ya. Arsenin, Methods of Solving Incorrect Problems [in Russian], Nauka, Moscow (1979).
2. V. A. Morozov, N. L. Gol'dman, and M. K. Samarin, "Methods of descriptive regularization and the quality of approximate solutions," Inzh.-Fiz. Zh., 33, No. 6, 1117-1123 (1977).
3. C. Boor, A Practical Guide to Spline Function, Springer-Verlag, New York (1978).
4. Yu. S. Zav'yalov, B. I. Kvasov, and V. L. Miroshnichenko, Methods of Spline Functions [in Russian], Nauka, Moscow (1980).
5. F. P. Vasil'ev, Numerical Methods of Solving Extremum Problems [in Russian], Nauka, Moscow (1980).
6. Yu. E. Voskoboinikov, "Construction of descriptive approximations of experimental data by B-splines," in: Algorithmic and instrumental means of information processing, Institut Teplofizika Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1981), pp. 36-46.
7. Yu. E. Voskoboinikov, "Criteria and algorithms for selecting parameters in smoothing by spline functions," in: Algorithms for Processing and Means of Automating Thermophysical Experiments, Institut Teplofiziki Sib. Otd. Akad. Nauk SSSR, Novosibirsk (1978), pp. 30-45.
8. G. H. Golub and M. Heath, "Generalized cross-validation as a method for choosing a good ridge parameter," Technometrics, 21, No. 2, 215-223 (1979).